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THREE-DIMENSIONAL ANALYSIS OF AN ELLIPSOIDAL INCLUSION IN A PIEZOELECTRIC MATERIAL

BIAO WANG
School of Astronautics, Harbin Institute of Technology, Harbin 150006, China

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Abstract—In this paper, the Green's function technique is used to develop a solution for an infinite, piezoelectric medium containing a piezoelectric, ellipsoidal inclusion. The coupled elastic and electric fields both inside the inclusion and on the boundary of the inclusion and matrix are obtained. These results are used to calculate the effective constants of piezoelectric composite materials. It is found that the coupled elastic and electric fields inside the inclusion stay uniform when the external elastic field and electric field are constant. As an example, the cylindrical inclusion is considered in detail and some formulae for calculating the effective constants of piezoelectric, unidirectional-fiber composites are obtained.

1. INTRODUCTION

Along with the widespread application of piezoelectric ceramics and piezoelectric composites, how to determine the effects of defects and inclusions on the properties of such materials becomes one of the most important problems in engineering. Generally, a piezoelectric ceramic material is a complex system composed of crystallites, grain boundaries and pores, and it may also contain many visible cracks perpendicular to the poling direction. The existence of these defects greatly affects the electric, dielectric, piezoelectric, elastic and mechanical properties of piezoelectric ceramics (Okazaki, 1985). In recent years, several types of PZT-polymer composites have been fabricated to improve the piezoelectric properties of poled PZT (lead zirconate titanate) ceramics (Rittenmyer *et al.*, 1985). How to predict the effective constants according to their constituent properties becomes a very important topic in the design of PZT-polymer composites, and the solution of all these problems relies on the analysis of the coupled elastic field and electric field of a typical inclusion in piezoelectric media. According to the author's knowledge, such a three-dimensional analysis is not available at present.

This research attempts to obtain the coupled elastic field and electric field of a piezoelectric, ellipsoidal inclusion in an infinite piezoelectric matrix. Two main difficulties exist in such analysis. One is that the piezoelectric materials are anisotropic, and the other difficulty is that the elastic fields and electric fields are coupled in such materials. In spite of these difficulties, the Green's function technique proved to be an efficient method to deal with such problems. By using the Green's function method, Kinoshita and Mura (1971) obtained the elastic field for an ellipsoidal inclusion in non-piezoelectric, anisotropic media, and Shintani and Minagawa (1988) have calculated the displacement and electric fields produced by moving dislocations in anisotropic, piezoelectric crystals. Zhou *et al.* (1986) proposed the multipole function representation and used the analogy theorem to obtain the elastoelectromagnetic field equations for a finite piezoelectric body with defects. Wang and Liu (1990) have obtained a general expression for the coupled elastic and electric fields of an ellipsoidal inclusion in a piezoelectric matrix based on the Green's function technique.

In this paper, the coupled elastic and electric fields inside an ellipsoidal inclusion and just outside the ellipsoidal inclusion are obtained, then these results are used to calculate the effective properties of the piezoelectric composites. As a simple, but important example, the elastic and electric fields of a cylindrical inclusion are investigated in detail, and some formulae for calculating the effective elastic, piezoelectric and dielectric constants of piezoelectric, unidirectional-fiber composites are obtained.

2. FORMULATIONS

If the free charges and body forces do not exist in a piezoelectric body, the static elastic and electric field equations can be written as (Maugin, 1988)

$$\partial_i D_i = 0, \tag{1}$$

$$\partial_i \sigma_{ij} = 0, \tag{2}$$

$$D_i = -a_{ij}\Phi_{,j} + e_{ikl}u_{k,l}, \tag{3}$$

$$\sigma_{ij} = C_{ijkl}u_{k,l} + e_{mij}\Phi_{,m}, \tag{4}$$

where **C** is the elastic moduli tensor, measured at zero strain, **e** is the piezoelectric moduli tensor and **a** is the permittivity of the dielectric material, and for transversely isotropic, piezoelectric, ceramic materials, they contain five, three and two independent constants, respectively. $\mathbf{\Phi}$ and **u** in eqns (2) and (3) are the electric potential and the elastic displacement. **D** and $\boldsymbol{\sigma}$ in eqns (1) and (2) are the electric displacement and the elastic stress tensor. Substitution of eqns (3) and (4) into eqns (1) and (2) yields

$$(C_{ijkl}u_{k,l})_{,i} + (e_{mij}\Phi_{,m})_{,i} = 0, \tag{5}$$

$$(e_{mij}u_{i,j})_{,m} - (a_{ml}\Phi_{,l})_{,m} = 0. \tag{6}$$

Consider an infinite piezoelectric body with the elastic moduli **C**, the piezoelectric moduli $\mathbf{e}^0$ and the dielectric permittivity $\mathbf{a}^0$ in which there is an inhomogeneous inclusion occupying a region Ω with constants **C**, **e** and **a**. By introducing the following notations:

$$C^1_{ijkl} = C_{ijkl} - C^0_{ijkl}, \tag{7}$$

$$e^1_{mij} = e_{mij} - e^0_{mij}, \tag{8}$$

$$a^1_{kl} = a_{kl} - a^0_{kl}, \tag{9}$$

the elastic, piezoelectric and dielectric constant tensors of the inhomogeneous medium can be written as

$$C_{ijkl}(\vec{x}) = C^0_{ijkl} + C^1_{ijkl}h(\vec{x}), \tag{10}$$

$$e_{mij}(\vec{x}) = e^0_{mij} + e^1_{mij}h(\vec{x}), \tag{11}$$

$$a_{kl}(\vec{x}) = a^0_{kl} + a^1_{kl}h(\vec{x}), \tag{12}$$

where $h(\vec{x})$ is the characteristic function and defined by

$$h(\vec{x}) = \begin{cases} 1, & \vec{x} \in \Omega \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Substitution of eqns (10), (11) and (12) into eqns (5) and (6) yields

$$C^0_{ijkl}u_{k,lj} + e^0_{mij}\Phi_{,mj} = -[C^1_{ijkl}u_{k,l}h(\vec{x})]_{,j} - [e^1_{mij}\Phi_{,m}h(\vec{x})]_{,j}, \tag{14}$$

$$e^0_{mij}u_{i,jm} - a^0_{ml}\Phi_{,lm} = -[e^1_{mij}u_{i,j}h(\vec{x})]_{,m} + [a^1_{ml}\Phi_{,l}h(\vec{x})]_{,m}. \tag{15}$$

By introducing the Green's functions $\mathbf{G}^1$, $\mathbf{G}^2$, $\mathbf{F}^1$, $\mathbf{F}^2$ as follows:

$$C^0_{ijkl}G^1_{kp,lj}+e^0_{mij}F^1_{p,mj} = -\delta_{ip}\delta(\vec{x}-\vec{x}'), \tag{16}$$

$$e^0_{jkl}G^1_{kp,lj}-a^0_{jk}F^1_{p,jk} = 0, \tag{17}$$

$$C^0_{ijkl}G^2_{k,lj}+e^0_{kij}F^2_{,kj} = 0, \tag{18}$$

$$e^0_{jkl}G^2_{k,lj}-a^0_{jk}F^2_{,jk} = -\delta(\vec{x}-\vec{x}'), \tag{19}$$

eqns (14) and (15) can be expressed in the form

$$\begin{aligned} u_m &= \int_V G^1_{mj}(\vec{x}-\vec{x}')[(C^1_{ijkl}u_{k,l}+e^1_{kij}\Phi_{,k})h(\vec{x}')]_{,i'}\,\mathrm{d}\vec{x}' \\ &\quad+\int_V G^2_m(\vec{x}-\vec{x}')[(e^1_{ikl}u_{k,l}-a^1_{il}\Phi_{,l})h(\vec{x}')]_{,i'}\,\mathrm{d}\vec{x}'+u^0_m \\ &= u^0_m+\int_\Omega G^1_{mj,i}(\vec{x}-\vec{x}')(C^1_{ijkl}u_{k,l}+e^1_{kij}\Phi_{,k})\,\mathrm{d}\vec{x}' \\ &\quad+\int_\Omega G^2_{m,i}(\vec{x}-\vec{x}')(e^1_{ikl}u_{k,l}-a^1_{il}\Phi_{,l})\,\mathrm{d}\vec{x}', \end{aligned} \tag{20}$$

$$\begin{aligned} \Phi &= \int_V F^1_j(\vec{x}-\vec{x}')[(C^1_{ijkl}u_{k,l}+e^1_{kij}\Phi_{,k})h(\vec{x}')]_{,i'}\,\mathrm{d}\vec{x}' \\ &\quad+\int_V F^2(\vec{x}-\vec{x}')[(e^1_{ikl}u_{k,l}-a^1_{il}\Phi_{,l})h(\vec{x}')]_{,i'}\,\mathrm{d}\vec{x}'+\Phi_0 \\ &= \Phi_0+\int_\Omega F^1_{j,i}(\vec{x}-\vec{x}')(C^1_{ijkl}u_{k,l}+e^1_{kij}\Phi_{,k})\,\mathrm{d}\vec{x}' \\ &\quad+\int_\Omega F^2_{,i}(\vec{x}-\vec{x}')(e^1_{ikl}u_{k,l}-a^1_{il}\Phi_{,l})\,\mathrm{d}\vec{x}', \end{aligned} \tag{21}$$

where u^0_m and Φ_0 are homogeneous solutions of eqns (14) and (15). In the derivation of eqns (20) and (21), the property of the generalized function $h(\vec{x})$ and the relation

$$G_{mj,i'}(\vec{x}-\vec{x}') = -G_{mj,i}(\vec{x}-\vec{x}') \tag{22}$$

is used.

By differentiating eqns (20) and (21), the equations of the elastic strain field and electric field can be obtained in the form

$$\begin{aligned} \varepsilon_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta}+u_{\beta,\alpha}) &= \varepsilon^0_{\alpha\beta}+\int_\Omega K^1_{\alpha\beta ij}(\vec{x}-\vec{x}')(C^1_{ij\gamma\rho}\varepsilon_{\gamma\rho}-e^1_{mij}E_m)\,\mathrm{d}\vec{x}' \\ &\quad+\int_\Omega K^2_{\alpha\beta i}(\vec{x}-\vec{x}')(e^1_{ikl}\varepsilon_{kl}+a^1_{il}E_l)\,\mathrm{d}\vec{x}, \end{aligned} \tag{23}$$

$$\begin{aligned} E_\alpha = -\Phi_{,\alpha} &= E^0_\alpha+\int_\Omega S^1_{\alpha ij}(\vec{x}-\vec{x}')(C^1_{ijkl}\varepsilon_{kl}-e^1_{kij}E_k)\,\mathrm{d}\vec{x}' \\ &\quad+\int_\Omega S^2_{\alpha i}(\vec{x}-\vec{x}')(e^1_{ikl}\varepsilon_{kl}+a^1_{il}E_l)\,\mathrm{d}\vec{x}', \end{aligned} \tag{24}$$

where

$$K^{1}_{\alpha\beta ij}(\vec{x}-\vec{x}') = \tfrac{1}{2}(G^{1}_{\alpha j,i\beta}+G^{1}_{\beta j,i\alpha}), \tag{25}$$

$$K^{2}_{\alpha\beta i}(\vec{x}-\vec{x}') = \tfrac{1}{2}(G^{2}_{\alpha,i\beta}+G^{2}_{\beta,i\alpha}), \tag{26}$$

$$S^{1}_{\alpha ij}(\vec{x}-\vec{x}') = -F^{1}_{j,i\alpha}, \tag{27}$$

$$S^{2}_{\alpha i}(\vec{x}-\vec{x}') = -F^{2}_{,i\alpha}. \tag{28}$$

In a general case, the explicit forms of the Green's functions $\mathbf{G}^1$, $\mathbf{G}^2$, $\mathbf{F}^1$ and $\mathbf{F}^2$ cannot be obtained from eqns (16), (17), (18) and (19). Whereas the Fourier transform of the Green's functions can be obtained easily, the Fourier transforms of eqns (16), (17), (18) and (19) are

$$C^{0}_{ijkl}\xi_l\xi_j G^{1T}_{kp}(\vec{\xi})+e^{0}_{kij}\xi_k\xi_j F^{1T}_{p}(\vec{\xi}) = \delta_{ip}, \tag{29}$$

$$e^{0}_{jkl}\xi_l\xi_j G^{1T}_{kp}(\vec{\xi})-\alpha^{0}_{jk}\xi_k\xi_j F^{1T}_{p}(\vec{\xi}) = 0, \tag{30}$$

$$C^{0}_{ijkl}\xi_l\xi_j G^{2T}_{k}(\vec{\xi})+e^{0}_{kij}\xi_k\xi_j F^{2T}(\vec{\xi}) = 0, \tag{31}$$

$$e^{0}_{jkl}\xi_l\xi_j G^{2T}_{k}(\vec{\xi})-\alpha^{0}_{jk}\xi_k\xi_j F^{2T}(\vec{\xi}) = 1, \tag{32}$$

where

$$G^{1}_{kp}(\vec{x}-\vec{x}') = \frac{1}{8\pi^3}\int G^{1T}_{kp}(\vec{\xi})\exp\,[i\vec{\xi}\cdot(\vec{x}-\vec{x}')]\,\mathrm{d}\vec{\xi}, \tag{33}$$

and G^{2T}_{k}, F^{1T}_{p} and F^{2T} can be determined similarly.

The eqns (29), (30), (31) and (32) can be expressed in the fourth-order matrix form as

$$\begin{bmatrix} C^{0}_{ijkl}\xi_l\xi_j & e^{0}_{kij}\xi_k\xi_j \\ e^{0}_{jkl}\xi_l\xi_j & -\alpha^{0}_{jk}\xi_j\xi_k \end{bmatrix}\begin{bmatrix} G^{1T}_{kp} & F^{1T}_{p} \\ G^{2T}_{k} & F^{2T} \end{bmatrix} = \begin{bmatrix} \delta_{ip} & 0 \\ 0 & 1 \end{bmatrix}, \tag{34}$$

from which we can find

$$G^{2T}_{k} = F^{1T}_{k}. \tag{35}$$

Substitution of eqn (33) into eqns (23) and (24) gives

$$\varepsilon_{\alpha\beta} = \varepsilon^{0}_{\alpha\beta} - \frac{1}{16\pi^3}\int_\Omega \mathrm{d}\vec{x}'\int (G^{1T}_{\alpha i}\xi_i\xi_\beta + G^{1T}_{\beta i}\xi_i\xi_\alpha)(C^{1}_{ijvp}\varepsilon_{vp} - e^{1}_{mij}E_m)$$
$$\times\exp\,[i\vec{\xi}\cdot(\vec{x}-\vec{x}')]\,\mathrm{d}\vec{\xi} - \frac{1}{16\pi^3}\int_\Omega \mathrm{d}\vec{x}'\int (G^{2T}_{\alpha}\xi_i\xi_\beta + G^{2T}_{\beta}\xi_i\xi_\alpha)$$
$$\times(e^{1}_{ikl}\varepsilon_{kl} + \alpha^{1}_{il}E_l)\exp\,[i\vec{\xi}\cdot(\vec{x}-\vec{x}')]\,\mathrm{d}\vec{\xi}, \tag{36}$$

$$E_\alpha = E^{0}_{\alpha} + \frac{1}{8\pi^3}\int_\Omega \mathrm{d}\vec{x}\int F^{1T}_{j}\xi_i\xi_\alpha(C^{1}_{ijkl}\varepsilon_{kl} - e^{1}_{kij}E_k)\exp\,[i\vec{\xi}\cdot(\vec{x}-\vec{x}')]\,\mathrm{d}\vec{\xi}$$
$$+\frac{1}{8\pi^3}\int_\Omega \mathrm{d}\vec{x}'\int F^{2T}\xi_i\xi_\alpha(e^{1}_{ikl}\varepsilon_{kl} + \alpha^{1}_{il}E_l)\exp\,[i\vec{\xi}\cdot(\vec{x}-\vec{x}')]\,\mathrm{d}\vec{\xi}. \tag{37}$$

In deriving eqns (36) and (37), the equation

$$G^{1}_{kp,ij}(\vec{x}-\vec{x}') = -\frac{1}{8\pi^3}\int G^{1T}_{kp}\xi_i\xi_j \exp[i\vec{\xi}\cdot(\vec{x}-\vec{x}')]\,d\vec{\xi} \tag{38}$$

has been used.

If the inclusion occupies an ellipsoidal sub-domain Ω, the integrals in eqns (36) and (37) can be further simplified by following the same procedure as Mura (1982) in deriving the elastic field of an anisotropic, ellipsoidal inclusion in non-piezoelectric media. We take the following integral in eqn (36) as an example:

$$I_{\alpha\beta} = \int d\vec{x}' \int G^{1T}_{\alpha j}\xi_i\xi_\beta(C^{1}_{ij\gamma\rho}\varepsilon_{\gamma\rho} - e^{1}_{mij}E_m)\exp[i\vec{\xi}\cdot(\vec{x}-\vec{x}')]\,d\vec{\xi}. \tag{39}$$

The integration with respect to $\vec{\xi}$-space is considered first. The volume element in $\vec{\xi}$-space, $d\vec{\xi}$ is

$$d\vec{\xi} = d\xi_1\,d\xi_2\,d\xi_3 = \xi^2\,d\xi\,dS(\vec{w}), \tag{40}$$

where

$$\xi = (\xi_1^2+\xi_2^2+\xi_3^2)^{1/2}, \quad w_i = \xi_i/\xi, \tag{41}$$

and $dS(\vec{w})$ is a surface element on the unit sphere S^2 in the $\vec{\xi}$-space. Then eqn (39) is written as

$$\begin{aligned}
I_{\alpha\beta} &= \int_\Omega d\vec{x}' \int_0^\infty \xi^2\,d\xi \int_{S^2} G^{1T}_{\alpha j}\xi_i\xi_\beta(C^{1}_{ij\gamma\rho}\varepsilon_{\gamma\rho} - e^{1}_{mij}E_m)\exp[i\xi\vec{w}\cdot(\vec{x}-\vec{x}')]\,dS(\vec{w})\\
&= \frac{1}{2}\int_\Omega d\vec{x}' \int_{-\infty}^\infty \xi^2\,d\xi \int_{S^2} G^{1T}_{\alpha j}\xi_i\xi_\beta(C^{1}_{ij\gamma\rho}\varepsilon_{\gamma\rho} - e^{1}_{mij}E_m)\exp[i\xi\vec{w}\cdot(\vec{x}-\vec{x}')]\,dS(\vec{w})\\
&= \frac{1}{2}\int_\Omega (C^{1}_{ij\gamma\rho}\varepsilon_{\gamma\rho} - e^{1}_{mij}E_m)\,d\vec{x}' \int_{S^2} \xi^2 G^{1T}_{\alpha j} w_i w_\beta\,dS(\vec{w}) \int_{-\infty}^\infty \xi^2 \exp[i\xi\vec{w}\cdot(\vec{x}-\vec{x}')]\,d\xi\\
&= -\pi \int_\Omega d\vec{x}' \int_{S^2} \xi^2 G^{1T}_{\alpha j} w_i w_\beta \delta''[\vec{w}\cdot(\vec{x}-\vec{x}')](C^{1}_{ij\gamma\rho}\varepsilon_{\gamma\rho} - e^{1}_{mij}E_m)\,dS(\vec{w}).
\end{aligned} \tag{42}$$

In the above integrals, the following equations

$$\xi^2 G^{1T}_{\alpha j} = G^{1T}_{\alpha j}(\vec{w}), \quad \int_{-\infty}^\infty \exp(i\xi\eta)\,d\xi = 2\pi\delta(\eta), \tag{43}$$

have been used.

If the crystalline directions of the anisotropic materials coincide with the principal axes of the ellipsoidal inclusion, the region Ω is expressed by

$$\frac{(x_1')^2}{a_1^2} + \frac{(x_2')^2}{a_2^2} + \frac{(x_3')^2}{a_3^2} = 1. \tag{44}$$

The following transformations of variables are used to simplify the calculating procedure:

$$x_1/a_1 = y_1,\quad x_2/a_2 = y_2,\quad x_3/a_3 = y_3,\quad x'_1/a_1 = y'_1,\quad x'_2/a_2 = y'_2,$$
$$x'_3/a_3 = y'_3,\quad a_1 w_1 = \rho_1,\quad a_2 w_2 = \rho_2,\quad a_3 w_3 = \rho_3,$$
$$a_1 w_1/\rho = \bar{w}_1,\quad a_2 w_2/\rho = \bar{w}_2,\quad a_3 w_3/\rho = \bar{w}_3,\quad \rho = (\rho_1^2+\rho_2^2+\rho_3^2)^{1/2}. \tag{45}$$

The volume element $d\vec{x}'$ is

$$d\vec{x}' = dx'_1\, dx'_2\, dx'_3 = a_1 a_2 a_3\, dy'_1\, dy'_2\, dy'_3 = a_1 a_2 a_3 r\, dr\, d\theta\, dz. \tag{46}$$

Substitution of eqns (45) and (46) into eqn (42) gives

$$I_{\alpha\beta} = -\pi a_1 a_2 a_3 \int_{-1}^{1} dz \int_0^{2\pi} d\Phi \int_0^R r\, dr \int_{S^2} G_{\alpha j}^{1T}(\vec{w}) w_i w_\beta \delta''(\rho \bar{w}_k y_k - \rho Z)$$
$$\times (C^1_{ijvp}\varepsilon_{vp} - e^1_{mij}E_m)\, dS(\vec{w}), \tag{47}$$

where $R = \sqrt{1-Z^2}$.

Let us consider the case when point $\vec{x}$ is inside the ellipsoid or point $\vec{y}$ is inside the unit sphere. Since the boundary values of the integration by parts vanish, after twice integrating by parts with respect to Z, eqn (47) becomes

$$I_{\alpha\beta} = -\pi a_1 a_2 a_3 \int_0^{2\pi} d\Phi \Bigg[\int_0^R r\, dr \frac{\partial^2}{\partial Z^2}(C^1_{ijvp}\varepsilon^I_{vp} - e^1_{mij}E^I_m)$$
$$- Z\left\{\frac{\partial}{\partial Z}(C^1_{ijvp}\varepsilon^I_{vp} - e^1_{mij}E^I_m)\right\}_{r=R} - Z\frac{\partial}{\partial Z}\{C^1_{ijvp}\varepsilon^I_{vp} - e^1_{mij}E^I_m\}_{r=R}$$
$$- \{C^1_{ijvp}\varepsilon^I_{vp} - e^1_{mij}E^I_m\}_{r=R}\Bigg]_{Z=\bar{w}_i y_i} \int_{S^2} G_{\alpha j}^{1T}(\vec{w}) w_i w_\beta \rho^{-3}\, dS(\vec{w}), \tag{48}$$

where the upper index I refers to the field values for interior points. It can be seen from (48) that if ε^I and E^I are constant, $I_{\alpha\beta}$ is also a constant, and determined by

$$I_{\alpha\beta} = 2\pi^2 a_1 a_2 a_3 (C^1_{ijvp}\varepsilon^I_{vp} - e^1_{mij}E^I_m) \int_{S^2} G_{\alpha j}^{1T}(\vec{w}) w_i w_\beta \rho^{-3}\, dS(\vec{w})$$
$$= 2\pi^2 (C^1_{ijvp}\varepsilon^I_{vp} - e^1_{mij}E^I_m) \int_{S^2} G_{\alpha j}^{1T}(\vec{w}) w_i w_\beta\, dS(\bar{w}_i), \tag{49}$$

where

$$dS(\bar{w}_i) = a_1 a_2 a_3 \rho^{-3}\, dS(\vec{w}). \tag{50}$$

The other integrals in eqns (36) and (37) can be coped with in the same manner as $I_{\alpha\beta}$. Since the solutions of eqns (36) and (37) are unique, we can conclude that if a piezoelectric, ellipsoidal inclusion in an infinite piezoelectric matrix is subjected to the uniform elastic field $\varepsilon^0_{\alpha\beta}$ and electric field E^0, the elastic field and the electric field inside the inclusion remain uniform. Such a conclusion is also well known in electricity (Maugin, 1988) and elasticity (Eshelby, 1957).

Furthermore, the surface element $dS(\bar{w}_i)$ can be written as

$$dS(\bar{w}_i) = d\bar{w}_3\, d\theta. \tag{51}$$

Then eqn (49) is written as

$$I_{\alpha\beta} = 2\pi^2(C^1_{ij\gamma\rho}\varepsilon^I_{\gamma\rho} - e^1_{mij}E^I_m)\int_{-1}^{1} d\bar{w}_3 \int_0^{2\pi} w_i w_\beta G^{1T}_{\alpha j}(\vec{w})\, d\theta \tag{52}$$

where

$$w_1 = \bar{w}_1/a_1, \quad w_2 = \bar{w}_2/a_2, \quad w_3 = \bar{w}_3/a_3, \tag{53}$$

and

$$\bar{w}_1 = (1-\bar{w}_3^2)^{1/2}\cos\theta, \quad \bar{w}_2 = (1-\bar{w}_3^2)^{1/2}\sin\theta, \quad \bar{w}_3 = \bar{w}_3. \tag{54}$$

By using eqn (52), eqn (36) and (37) can be expressed as

$$\varepsilon^I_{\alpha\beta} = \varepsilon^0_{\alpha\beta} - \frac{1}{8\pi}(N^1_{\alpha i\beta j} + N^1_{\beta i\alpha j})(C^1_{ij\gamma\rho}\varepsilon^I_{\gamma\rho} - e^1_{mij}E^I_m) - \frac{1}{8\pi}(N^2_{\alpha\beta i} + N^2_{\beta\alpha i})(e^1_{ikl}\varepsilon^I_{kl} + a^1_{il}E^I_l), \tag{55}$$

$$E^I_\alpha = E^0_\alpha + \frac{1}{4\pi}N^2_{ji\alpha}(C^1_{ijkl}\varepsilon^I_{kl} - e^1_{kij}E^I_k) + \frac{1}{4\pi}N^3_{\alpha i}(e^1_{ikl}\varepsilon^I_{kl} + a^1_{il}E^I_l), \tag{56}$$

where

$$N^1_{ijkl} = \int_{-1}^{1} d\bar{w}_3 \int_0^{2\pi} G^{1T}_{ij}(\vec{w})w_k w_l\, d\theta, \tag{57}$$

$$N^2_{ijk} = \int_{-1}^{1} d\bar{w}_3 \int_0^{2\pi} G^{2T}_i(\vec{w})w_j w_k\, d\theta, \tag{58}$$

$$N^3_{ij} = \int_{-1}^{1} d\bar{w}_3 \int_0^{2\pi} F^{2T}(\vec{w})w_i w_j\, d\theta. \tag{59}$$

If the matrix is a transversely isotropic piezoelectric material (with X_3 in the poling direction), the non-zero components of $\mathbf{N}^1$, $\mathbf{N}^2$, $\mathbf{N}^3$ are shown in the Appendix.

In the same manner, the coupled strain and electric fields outside the inclusion can be obtained through eqns (36) and (37). The jumps of the strain field and the electric field on the inclusion boundary are given by

$$\begin{aligned}[\varepsilon_{\alpha\beta}] &= \varepsilon^E_{\alpha\beta} - \varepsilon^I_{\alpha\beta} \\ &= G^{1T}_{\alpha i}(\vec{n})[n_j C^1_{ijkl}\varepsilon^I_{kl}n_\beta - n_j e^1_{mij}E^I_m n_\beta] \\ &\quad + G^{2T}_\alpha(\vec{n})[n_m e^1_{mij}\varepsilon^I_{ij}n_\beta + n_m a^1_{ml}E^I_l n_\beta],\end{aligned} \tag{60}$$

$$\begin{aligned}[E_\alpha] &= E^E_\alpha - E^I_\alpha \\ &= -G^{2T}_i(\vec{n})[n_j C^1_{ijkl}\varepsilon^I_{kl}n_\alpha - n_j e^1_{mij}E^I_m n_\alpha] \\ &\quad - F^{2T}(\vec{n})[n_m e^1_{mij}\varepsilon^I_{ij}n_\alpha + n_m a^1_{ml}E^I_l n_\alpha],\end{aligned} \tag{61}$$

where $\boldsymbol{\varepsilon}^E$ and $\mathbf{E}^E$ are the strain field and the electric field just outside the inclusion, and $G^{1T}_{\alpha i}(\vec{n})$, $G^{2T}_\alpha(\vec{n})$ and $F^{2T}(\vec{n})$ are determined by eqn (34) with the replacement of $\vec{\xi}$ by $\vec{n}$, here $\vec{n}$ is the outward normal on the inclusion boundary.

Equations (60) and (61) can also be obtained directly through the boundary condition between the inclusion and the matrix as follows:

(a) Elastic field: The displacement and the interfacial traction across the boundary must be continuous, that is

$$u_i^I = u_i^E, \tag{62}$$

$$n_j\sigma_{ij}^I = n_i\sigma_{ij}^E. \tag{63}$$

From eqn (62), we know

$$[\varepsilon_{ij}] = \varepsilon_{ij}^E - \varepsilon_{ij}^I = \Delta_i n_j \tag{64}$$

where Δ_i is the proportionality constant to be determined.

(b) Electric field: The electric field and electric displacement across the boundary should satisfy the following conditions

$$\vec{n} \times (\vec{E}^E - \vec{E}^I) = 0, \tag{65}$$

$$n_i(D_i^E - D_i^I) = 0. \tag{66}$$

From eqn (65), we know

$$[E_i] = E_i^E - E_i^I = \lambda n_i. \tag{67}$$

Substituting the constitutive eqn (3) into eqn (63), one obtains

$$n_i\{c_{ijkl}^0[\varepsilon_{kl}] - c_{ijkl}^1\varepsilon_{kl}^I - e_{mij}^0[E_m] + e_{mij}^1 E_m^I\} = 0. \tag{68}$$

Substitution of eqn (4) into eqn (66) yields

$$n_k\{a_{kl}^0[E_l] - a_{kl}^1 E_l^I + e_{kij}^0[\varepsilon_{ij}] - e_{kij}^1\varepsilon_{ij}^I\} = 0. \tag{69}$$

Substitution of eqns (64) and (67) into eqns (68) and (69) gives

$$n_i C_{ijkl}^0 n_l \Delta_k - n_i e_{mij}^0 n_m \lambda = n_i c_{ijkl}^1 \varepsilon_{kl}^I - n_i e_{mij}^1 E_m^I, \tag{70}$$

$$n_k a_{kl}^0 n_l \lambda + n_k e_{kij}^0 n_i \Delta_j = n_k a_{kl}^1 E_l^I + n_k e_{kij}^1 \varepsilon_{ij}^I, \tag{71}$$

from which we obtain

$$\Delta_a = G_{aj}^{1T}(\vec{n})[n_i C_{ijkl}^1 \varepsilon_{kl}^I - n_i e_{mij}^1 E_m^I] + G_a^{2T}(\vec{n})[n_k a_{kl}^1 E_l^I + n_k e_{kij}^1 \varepsilon_{ij}^I], \tag{72}$$

$$\lambda = -G_j^{2T}(\vec{n})[n_i C_{ijkl}^1 \varepsilon_{kl}^I - n_i e_{mij}^1 E_m^I] - F^{2T}(\vec{n})[n_m e_{mij}^1 \varepsilon_{ij}^I + n_m a_{ml}^1 E_l^I]. \tag{73}$$

Substitution of eqn (72) into eqn (64) and eqn (73) into eqn (67) gives the same results as eqns (60) and (61). The coupled elastic and electric fields just outside an inclusion can be evaluated from eqns (60) and (61) when ε_{ij}^I and E_i^I are known.

If the strain field and electric field inside an inclusion are known, the effective elastic, piezoelectric and dielectric constants of piezoelectric composites can be calculated as follows:

Definition. The effective elastic, piezoelectric and dielectric constants of piezoelectric composites, C_{ijkl}^*, E_{mij}^* and a_{kl}^* are defined by the following equations:

$$\langle\sigma_{ij}\rangle = C_{ijkl}^*\langle\varepsilon_{kl}\rangle - e_{mij}^*\langle E_m\rangle, \tag{74}$$

$$\langle D_k\rangle = a_{kl}^*\langle E_l\rangle + e_{kij}^*\langle\varepsilon_{ij}\rangle, \tag{75}$$

where the symbol $\langle\ \rangle$ denotes the volume average.

From eqn (74), it can be written that

$$
\begin{aligned}
\langle \sigma_{ij} \rangle &= C^*_{ijkl} \langle \varepsilon_{kl} \rangle - e^*_{mij} \langle E_m \rangle, \\
&= \frac{1}{v} \int_v \sigma_{ij} \, dv \\
&= \frac{1}{v} \left[\int_{v_I} (C_{ijkl} \varepsilon^I_{kl} - e_{mij} E^I_m) \, dv + \int_{v_m} (C^0_{ijkl} \varepsilon_{kl} - e^0_{mij} E_m) \, dv \right] \\
&= \frac{1}{v} \left[\int_{v_I} (C_{ijkl} \varepsilon^I_{kl} - e_{mij} E^I_m) \, dv + \int_{v_m} (C^0_{ijkl} \varepsilon_{kl} - e^0_{mij} E_m) \, dv \right. \\
&\quad \left. + \int_{v_I} (C^0_{ijkl} \varepsilon_{kl} - e^0_{mij} E_m) \, dv - \int_{v_I} (C^0_{ijkl} \varepsilon_{kl} - e^0_{mij} E_m) \, dv \right] \\
&= C^0_{ijkl} \langle \varepsilon_{kl} \rangle - e^0_{mij} \langle E_m \rangle + v_f C^1_{ijkl} \varepsilon^I_{kl} - v_f e^1_{mij} E^I_m, \qquad (76)
\end{aligned}
$$

where v_f is the volume fraction of inclusions. In the derivation of eqn (76), the interactions between inclusions are neglected. In the same manner, from eqn (75), we obtain

$$
\begin{aligned}
\langle D_k \rangle &= a^*_{kl} \langle E_l \rangle + e^*_{kij} \langle \varepsilon_{ij} \rangle \\
&= a^0_{kl} \langle E_l \rangle + e^0_{kij} \langle \varepsilon_{ij} \rangle + v_f a^1_{kl} E^I_l + v_f e^1_{kij} \varepsilon^I_{ij}. \qquad (77)
\end{aligned}
$$

3. CYLINDRICAL INCLUSION

As an important example, the cylindrical inclusion is considered in detail. Both the inclusion and the matrix are assumed to be transversely isotropic. Their non-zero constants are

$$
\begin{aligned}
&C^0_{11} = C^0_{22}, \quad C^0_{13} = C^0_{23}, \quad C^0_{33}, \quad C^0_{44} = C^0_{55}, \quad C^0_{66} = \tfrac{1}{2}(C^0_{11} - C^0_{12}), \\
&e^0_{31} = e^0_{32}, \quad e^0_{33}, \quad e^0_{15} = e^0_{24}, \quad a^0_{11} = a^0_{22}, \quad a^0_{33}, \qquad (78)
\end{aligned}
$$

and $w_1 = \cos\theta$, $w_2 = \sin\theta$ and $\omega_3 = 0$, where 3-axis is the symmetric axis. In such cases, from eqn (34), one obtains

$$
G^{1T}_{11}(\vec{w}) = \frac{2C^0_{11} \sin^2\theta + (C^0_{11} - C^0_{12}) \cos^2\theta}{C^0_{11}(C^0_{11} - C^0_{12})} a^2, \qquad (79)
$$

$$
G^{1T}_{12}(\vec{w}) = G^{1T}_{21}(\vec{w}) = -\frac{(C^0_{11} + C^0_{12}) \sin\theta \cos\theta}{C^0_{11}(C^0_{11} - C^0_{12})} a^2, \qquad (80)
$$

$$
G^{1T}_{22}(\vec{w}) = \frac{2C^0_{11} \cos^2\theta + (C^0_{11} - C^0_{12}) \sin^2\theta}{C^0_{11}(C^0_{11} - C^0_{12})} a^2, \qquad (81)
$$

$$
G^{1T}_{33}(\vec{w}) = \frac{a^0_{11}}{C^0_{44} a^0_{11} + (e^0_{15})^2} a^2, \qquad (82)
$$

where eqns (53) and (54) are used, and the other components of $G^{1T}_{ij}(\vec{w})$ are zero, and a is the radius of the cylindrical inclusion cross-section

$$
\begin{aligned}
&G^{2T}_1(\vec{w}) = G^{2T}_2(\vec{w}) = 0, \\
&G^{2T}_3(\vec{w}) = \frac{e^0_{15} a^2}{C^0_{44} a^0_{11} + (e^0_{15})^2}, \qquad (83)
\end{aligned}
$$

$$F^{2T}(\vec{w}) = -\frac{C_{44}^0 a^2}{C_{44}^0 a_{11}^0 + (e_{15}^0)^2}. \tag{84}$$

Substitution of eqns (78)–(84) into eqns (57), (58) and (59) yields the non-zero components of $\mathbf{N}^1$, $\mathbf{N}^2$, $\mathbf{N}^3$ as:

$$N_{1111}^1 = N_{2222}^1 = \frac{5C_{11}^0 - 3C_{12}^0}{2C_{11}^0(C_{11}^0 - C_{12}^0)}\pi,$$

$$N_{1122}^1 = N_{2211}^1 = \frac{7C_{11}^0 - C_{12}^0}{2C_{11}^0(C_{11}^0 - C_{12}^0)}\pi,$$

$$N_{3311}^1 = N_{3322}^1 = \frac{2\pi a_{11}^0}{C_{44}^0 a_{11}^0 + (e_{15}^0)^2},$$

$$N_{1212}^1 = N_{2121}^1 = N_{2112}^1 = N_{1221}^1 = -\frac{C_{11}^0 + C_{12}^0}{2C_{11}^0(C_{11}^0 - C_{12}^0)}\pi, \tag{85}$$

$$N_{311}^2 = N_{322}^2 = \frac{2\pi e_{15}^0}{C_{44}^0 a_{11}^0 + (e_{15}^0)^2}. \tag{86}$$

$$N_{11}^3 = N_{22}^3 = -\frac{2C_{44}^0 \pi}{C_{44}^0 a_{11}^0 + (e_{15}^0)^2}. \tag{87}$$

Substituting eqns (85), (86) and (87) into eqns (55) and (56), one obtains the equations for determining the coupled strain and electric fields inside a cylindrical inclusion in the form of:

$$\varepsilon_{11}^I = \varepsilon_{11}^0 - \frac{5C_{11}^0 - 3C_{12}^0}{8C_{11}^0(C_{11}^0 - C_{12}^0)}(C_{11}^1\varepsilon_{11}^I + C_{12}^1\varepsilon_{22}^I + C_{13}^1\varepsilon_{33}^I) + \frac{C_{11}^0 + C_{12}^0}{8C_{11}^0(C_{11}^0 - C_{12}^0)}(C_{12}^1\varepsilon_{11}^I + C_{22}^1\varepsilon_{22}^I + C_{23}^1\varepsilon_{33}^I) + \frac{e_{31}^1}{2C_{11}^0}E_3^I, \tag{88}$$

$$\varepsilon_{22}^I = \varepsilon_{22}^0 + \frac{C_{11}^0 + C_{12}^0}{8C_{11}^0(C_{11}^0 - C_{12}^0)}(C_{11}^1\varepsilon_{11}^I + C_{12}^1\varepsilon_{22}^I + C_{13}^1\varepsilon_{33}^I) - \frac{5C_{11}^0 - 3C_{12}^0}{8C_{11}^0(C_{11}^0 - C_{12}^0)}(C_{12}^1\varepsilon_{11}^I + C_{22}^1\varepsilon_{22}^I + C_{23}^1\varepsilon_{33}^I) + \frac{e_{31}^1}{2C_{11}^0}E_3^I, \tag{89}$$

$$\varepsilon_{33}^I = \varepsilon_{33}^0, \tag{90}$$

$$\varepsilon_{23}^I = \frac{4[C_{44}^0 a_{11}^0 + (e_{15}^0)^2]\varepsilon_{23}^0 + (a_{11}^0 e_{24}^1 - a_{22}^1 e_{15}^0)E_2^I}{4[C_{44}^0 a_{11}^0 + (e_{15}^0)^2] + 2e_{15}^0 e_{24}^1 + 2a_{11}^0 C_{44}^1}, \tag{91}$$

$$\varepsilon_{13}^I = \frac{4[C_{44}^0 a_{11}^0 + (e_{15}^0)^2]\varepsilon_{13}^0 + (a_{11}^0 e_{24}^1 - a_{22}^1 e_{15}^0)E_1^I}{4[C_{44}^0 a_{11}^0 + (e_{15}^0)^2] + 2e_{15}^0 e_{24}^1 + 2a_{11}^0 C_{44}^1}, \tag{92}$$

$$\varepsilon_{12}^I = \left[1 + \frac{3C_{11}^0 - C_{12}^0}{2C_{11}^0(C_{11}^0 - C_{12}^0)}C_{66}^1\right]^{-1}\varepsilon_{12}^0, \tag{93}$$

$$E_1^I = \frac{2[C_{44}^0 a_{11}^0 + (e_{15}^0)^2]E_1^0 + 2(e_{15}^0 C_{55}^1 - e_{15}^1 C_{44}^0)\varepsilon_{13}^I}{2[C_{44}^0 a_{11}^0 + (e_{15}^0)^2] + e_{15}^0 e_{15}^1 + a_{11}^1 C_{44}^0}. \tag{94}$$

$$E_2^f = \frac{2[C_{44}^0 a_{11}^0 + (e_{15}^0)^2]E_2^0 + 2(e_{15}^0 C_{44}^1 - e_{24}^1 C_{44}^0)\varepsilon_{23}^f}{2[C_{44}^0 a_{11}^0 + (e_{15}^0)^2] + C_{44}^0 a_{11}^1 + e_{15}^0 e_{15}^1}, \tag{95}$$

$$E_3^f = E_3^0. \tag{96}$$

To obtain the effective elastic, piezoelectric and dielectric constants of unidirectional fiber composites, the following procedures are taken:

(a) Only $\varepsilon_{33}^0 \neq 0$.
From eqns (88) and (89), one obtains

$$\varepsilon_{11}^f = \varepsilon_{22}^f = -\frac{C_{13}^1}{2C_{11}^0 + C_{11}^1 + C_{12}^1}\varepsilon_{33}^0. \tag{97}$$

According to eqn (76),

$$\begin{aligned}\langle\sigma_{33}\rangle &= C_{3333}^* \varepsilon_{33}^0 \\ &= C_{33}^0 \varepsilon_{33}^0 + v_f C_{31}^1 \varepsilon_{11}^f + v_f C_{32}^1 \varepsilon_{22}^f + v_f C_{33}^1 \varepsilon_{33}^f,\end{aligned} \tag{98}$$

from which one obtains

$$C_{33}^* = C_{33}^0 + v_f C_{33}^1 - \frac{2v_f (C_{13}^1)^2}{2C_{11}^0 + C_{11}^1 + C_{12}^1}. \tag{99}$$

In the same manner, one obtains

$$C_{13}^* = C_{13}^0 + v_f C_{13}^1 - \frac{v_f C_{13}^1 (C_{11}^1 + C_{12}^1)}{2C_{11}^0 + C_{11}^1 + C_{12}^1}. \tag{100}$$

(b) $\varepsilon_{11}^0 = \varepsilon_{22}^0 \neq 0$, and $\varepsilon_{12}^0 \neq 0$.
From eqns (88) and (89), one obtains

$$\varepsilon_{11}^f = \varepsilon_{22}^f = \frac{2C_{11}^0}{2C_{11}^0 + C_{11}^1 + C_{12}^1}\varepsilon_{11}^0, \tag{101}$$

and ε_{12}^1 is given by eqn (93). Therefore

$$\begin{aligned}\langle\sigma_{11}\rangle &= C_{11}^* \varepsilon_{11}^0 + C_{12}^* \varepsilon_{22}^0 \\ &= C_{11}^0 \varepsilon_{11}^0 + C_{12}^0 \varepsilon_{22}^0 + v_f C_{11}^1 \varepsilon_{11}^f + v_f C_{12}^1 \varepsilon_{22}^f,\end{aligned} \tag{102}$$

$$\begin{aligned}\langle\sigma_{12}\rangle &= 2C_{66}^* \varepsilon_{12}^0 \\ &= 2C_{66}^0 \varepsilon_{12}^0 + 2v_f C_{66}^1 \varepsilon_{12}^f,\end{aligned} \tag{103}$$

and one can obtain

$$C_{11}^* + C_{12}^* = C_{11}^0 + C_{12}^0 + 2v_f (C_{11}^1 + C_{12}^1) \times \frac{C_{11}^0}{2C_{11}^0 + C_{11}^1 + C_{12}^1}, \tag{104}$$

and

$$\begin{aligned}C_{11}^* - C_{12}^* &= 2C_{66}^* \\ &= 2C_{66}^0 + 2v_f C_{66}^1 \left[1 + \frac{3C_{11}^0 - C_{12}^0}{2C_{11}^0 (C_{11}^0 - C_{12}^0)} C_{66}^1\right]^{-1}.\end{aligned} \tag{105}$$

(c) $\varepsilon_{23}^0 \neq 0$.

In such a case, from eqns (91) and (95), one obtains

$$\varepsilon^{I}_{23} = 4\left[4[a^0_{11}C^0_{44}+(e^0_{15})^2]+2a^0_{11}C^1_{44}+2e^0_{15}e^1_{24}-\frac{2(a^0_{11}e^1_{24}-a^1_{22}e^0_{15})(e^0_{15}C^1_{55}-C^0_{44}e^1_{15})}{2[a^0_{11}C^0_{44}+(e^0_{15})^2]+C^0_{44}a^1_{11}+e^0_{15}e^1_{15}}\right]^{-1}$$
$$\times [a^0_{11}C^0_{44}+(e^0_{15})^2]\varepsilon^0_{23}, \quad (106)$$

$$E^I_2 = \frac{2(e^0_{15}C^1_{55}-C^0_{44}e^1_{15})}{2[a^0_{11}C^0_{44}+(e^0_{15})^2]+C^0_{44}\alpha^1_{11}+e^0_{15}e^1_{15}}\varepsilon^I_{23}. \quad (107)$$

$$\begin{aligned}\langle\sigma_{23}\rangle &= 2C^*_{2323}\varepsilon^0_{23}\\ &= 2C^0_{44}\varepsilon^0_{23}+2v_fC^1_{44}\varepsilon^I_{23}-v_fe^1_{24}E^I_2. \end{aligned}\quad (108)$$

Substitution of eqns (106) and (107) into eqn (108) gives C^*_{44}.

According to eqn (77), the effective piezoelectric constant e^*_{24} can be obtained from the following equation:

$$\begin{aligned}\langle D_2\rangle &= e^*_{24}\varepsilon^0_{23}\\ &= e^0_{24}\varepsilon^0_{23}+v_fa^1_{22}E^I_2+v_fe^1_{24}\varepsilon^I_{23}. \end{aligned}\quad (109)$$

(d) $E^0_3 \neq 0$.

From eqns (96) and (76), one obtains

$$\begin{aligned}\langle\sigma_{33}\rangle &= -e^*_{33}E^0_3\\ &= -e^0_{33}E^0_3-v_fe^1_{33}E^0_3+2v_fC^1_{13}\varepsilon^I_{11}, \end{aligned}\quad (110)$$

where ε^1_{11} is determined by equations (88) and (89) as

$$\varepsilon^I_{11} = \varepsilon^I_{22} = \frac{e^1_{31}E^0_3}{2C^0_{11}+C^1_{11}+C^1_{12}}. \quad (111)$$

Therefore e^*_{33} is given by

$$e^*_{33} = v_fe_{33}+(1-v_f)e^0_{33}-2v_f\frac{C^1_{13}e^1_{31}}{2C^0_{11}+C^1_{11}+C^1_{12}}. \quad (112)$$

$$\begin{aligned}\langle\sigma_{11}\rangle &= -e^*_{31}E^0_3\\ &= -e^0_{31}E^0_3-v_fe^1_{31}E^0_3+v_f(C^1_{11}+C^1_{12})\varepsilon^I_{11}. \end{aligned}\quad (113)$$

Substitution of eqn (111) into eqn (113) gives

$$e^*_{31} = v_fe_{31}+(1-v_f)e^0_{31}-v_f\frac{(C^1_{11}+C^1_{12})e^1_{31}}{2C^0_{11}+C^1_{11}+C^1_{12}}. \quad (114)$$

From eqn (77), we know

$$\begin{aligned}\langle D_3\rangle &= a^*_{33}E^0_3\\ &= a^0_{33}E^0_3+v_fa^1_{33}E^0_3+2v_fe^1_{31}\varepsilon^I_{11}. \end{aligned}\quad (115)$$

Substitution of eqn (111) into eqn (114) yields

Table 1. Materials properties of a piezoelectric ceramic PZT-6B and polymer

	Elastic stiffnesses (10^{10} N m^{-2})					Piezoelectric coefficients (c m^{-2})			Dielectric constants (10^{10} c vm^{-1})	
	C_{11}	C_{33}	C_{44}	C_{12}	C_{13}	e_{31}	e_{33}	e_{15}	α_{11}	α_{33}
PZT-6B	16.8	16.3	2.71	6.0	6.0	−0.9	7.1	4.6	36	34
Polymer	0.45	0.45	0.11	0.24	0.24	0	0	0	0	0

$$a^*_{33} = a^0_{33} + v_f a^1_{33} + 2v_f \frac{(e^1_{31})^2}{2C^0_{11} + C^1_{11} + C^1_{12}}. \tag{116}$$

(e) $E^0_1 \neq 0$.

In such a case, a^*_{11} can be determined from the following equation:

$$\langle D_1 \rangle = a^*_{11} E^0_1$$
$$= a^0_{11} E^0_1 + v_f a^1_{11} E^I_1 + v_f e^1_{15} \varepsilon^I_{13}, \tag{117}$$

where E^I_1 and ε^I_{13} can be obtained through eqns (92) and (94).

Although every effective constant of piezoelectric, unidirectional-fiber composite has been obtained, one has to bear in mind that all these analyses have neglected the interactions between inclusions. To obtain the more accurate results, one can use some approximate methods, such as the self-consistent scheme, etc. to consider the interactions.

As an example, a piezoelectric ceramic PZT-6B which contains unidirectional polymer fibers along the poling axis (X_3-axis) is considered in detail. The engineering material constants for PZT-6B and the polymer are listed in Table 1. (Shindo and Ozawa, 1990.) The polymer is assumed to be an isotropic material with negligible piezoelectric constants and dielectric constants.

The most important constants of the composite in engineering are e^*_{33}, α^*_3 and the hydrostatic coefficient e^*_h ($= e^*_{33} + 2e^*_{31}$). By using eqns (112), (114) and (116), they are obtained and shown in Fig. 1 versus the volume fraction of polymer.

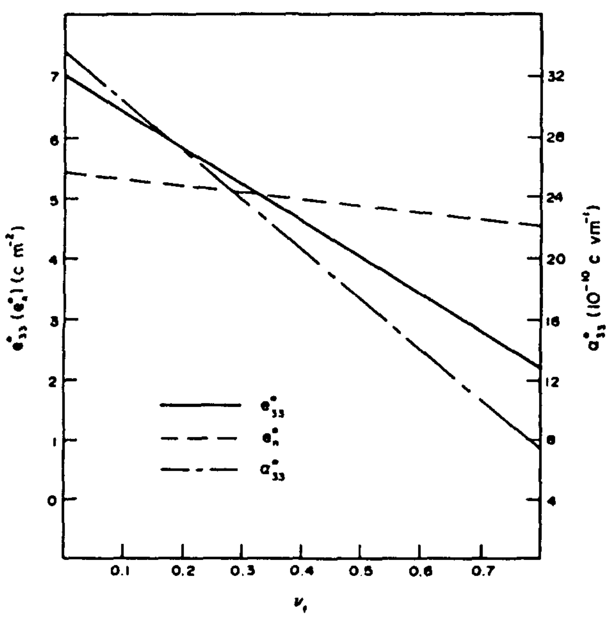

Fig. 1. The effective constants versus the volume fraction of polymer.

4. CONCLUDING REMARKS

In this paper, a solution is developed for an infinite, piezoelectric medium containing a piezoelectric, ellipsoidal inclusion. The coupled elastic and electric fields inside the inclusion are obtained, and it is found that they remain uniform when the external strain and electric fields are constant, which has again confirmed Eshelby's proposition. The coupled elastic and electric fields on the boundary of the inclusion and matrix are also obtained. By neglecting the interactions among inclusions, the above results are used to derive the effective constants of piezoelectric composite materials. As an example, the cylindrical inclusion is considered in detail. In such a case, the explicit form of the solution is obtained, and some formulae for calculating the effective constants of piezoelectric, unidirectional-fiber composite are derived. It is found that the commonly-used rule of mixture for determining the effective dielectric and piezoelectric constants along the poling axis is not true due to the coupled effects.

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APPENDIX: THE NON-ZERO COMPONENTS OF N^1, N^2, N^3, FOR A SPHEROIDAL INCLUSION IN A TRANSVERSELY ISOTROPIC, PIEZOELECTRIC MATRIX

For transversely isotropic, piezoelectric matrix, the non-zero material constants are shown in eqn (78). Therefore

$$N^1_{1111} = N^1_{2222} = 2\beta^2 \int_0^1 (1-\omega_3^2)\,\mathrm{d}\omega_3 \int_0^{2\pi} \frac{N^B_{11}\cos^4\theta + N^C_{11}\cos^2\theta}{A\cos^4\theta + B\cos^2\theta + C}\,\mathrm{d}\theta, \tag{A1}$$

$$N^1_{3333} = 2\int_0^1 \omega_3^2\,\mathrm{d}\omega_3 \int_0^{2\pi} \frac{N^A_{33}\cos^4\theta + N^B_{33}\cos^2\theta + N^C_{33}}{A\cos^4\theta + B\cos^2\theta + C}\,\mathrm{d}\theta, \tag{A2}$$

$$N^1_{1122} = N^1_{2211} = 2\beta^2 \int_0^1 (1-\omega_3^2)\,\mathrm{d}\omega_3 \int_0^{2\pi} \frac{N^A_{22}\cos^4\theta + N^B_{22}\cos^2\theta}{A\cos^4\theta + B\cos^2\theta + C}\,\mathrm{d}\theta, \tag{A3}$$

$$N^1_{1133} = N^1_{2233} = 2\int_0^1 \omega_3^2\,\mathrm{d}\omega_3 \int_0^{2\pi} \frac{N^B_{11}\cos^2\theta + N^C_{11}}{A\cos^4\theta + B\cos^2\theta + C}\,\mathrm{d}\theta, \tag{A4}$$

$$N^1_{3311} = N^1_{3322} = 2\beta^2 \int_0^1 (1-\omega_3^2)\,\mathrm{d}\omega_3 \int_0^{2\pi} \frac{\cos^2\theta\,(N^A_{33}\cos^4\theta + N^B_{33}\cos^2\theta + N^C_{33})}{A\cos^4\theta + B\cos^2\theta + C}\,\mathrm{d}\theta, \tag{A5}$$

$$N^1_{1212} = 2\beta^2 \int_0^1 (1-\omega_3^2)\,\mathrm{d}\omega_3 \int_0^{2\pi} \frac{N^B_{12}\sin^2\theta\cos^2\theta}{A\cos^4\theta + B\cos^2\theta + C}\,\mathrm{d}\theta, \tag{A6}$$

$$N^{1}_{1313} = N^{1}_{2323} = 2\beta \int_0^1 \omega_3(1-\omega_3^2)^{1/2}\,\mathrm{d}\omega_3 \int_0^{2\pi} \frac{N^{A}_{13}\cos^4\theta + N^{B}_{13}\cos^2\theta}{A\cos^4\theta + B\cos^2\theta + C}\,\mathrm{d}\theta, \tag{A7}$$

$$N^{2}_{113} = N^{2}_{223} = 2\beta \int_0^1 \omega_3(1-\omega_3^2)^{1/2}\,\mathrm{d}\omega_3 \int_0^{2\pi} \frac{N^{A}_{14}\cos^4\theta + N^{B}_{14}\cos^2\theta}{A\cos^4\theta + B\cos^2\theta + C}\,\mathrm{d}\theta, \tag{A8}$$

$$N^{2}_{311} = N^{2}_{322} = 2\beta^2 \int_0^1 (1-\omega_3^2)\,\mathrm{d}\omega_3 \int_0^{2\pi} \frac{\cos^2\theta\,(N^{A}_{34}\cos^4\theta + N^{B}_{34}\cos^2\theta + N^{C}_{34})}{A\cos^4\theta + B\cos^2\theta + C}\,\mathrm{d}\theta, \tag{A9}$$

$$N^{2}_{333} = 2 \int_0^1 \omega_3^2\,\mathrm{d}\omega_3 \int_0^{2\pi} \frac{N^{A}_{34}\cos^4\theta + N^{B}_{34}\cos^2\theta + N^{C}_{34}}{A\cos^4\theta + B\cos^2\theta + C}\,\mathrm{d}\theta, \tag{A10}$$

$$N^{3}_{11} = 2\beta^2 \int_0^1 (1-\omega_3^2)\,\mathrm{d}\omega_3 \int_0^{2\pi} \frac{\cos^2\theta\,(N^{A}_{44}\cos^4\theta + N^{B}_{44}\cos^2\theta + N^{C}_{44})}{A\cos^4\theta + B\cos^2\theta + C} \tag{A11}$$

$$N^{3}_{33} = 2 \int_0^1 \omega_3^2\,\mathrm{d}\omega_3 \int_0^{2\pi} \frac{N^{A}_{44}\cos^4\theta + N^{B}_{44}\cos^2\theta + N^{C}_{44}}{A\cos^4\theta + B\cos^2\theta + C}\,\mathrm{d}\theta, \tag{A12}$$

where

$$\begin{aligned} A = {} & a_{11}b_{21}c_3d_4 - a_{11}b_{21}c_4d_3 + a_{21}^2c_3d_4 - a_{21}^2c_4d_3 + a_{11}d_4b_{31}^2 - a_{11}b_{31}b_{41}d_{31} - a_{21}a_{31}b_{31}d_4 \\ & + a_{21}a_{31}b_{41}d_3 - a_{11}b_{31}b_{41}c_4 + a_{11}c_3b_{41}^2 + c_4a_{21}a_{41}b_{31} - c_3a_{21}a_{41}b_{41} - a_{21}a_{31}b_{31}d_4 \\ & + a_{21}a_{41}b_{31}d_3 - a_{31}^2b_{21}d_4 + a_{11}a_{41}b_{21}d_3 + a_{21}a_{31}b_{41}c_4 - a_{21}a_{41}b_{41}c_3 + a_{31}a_{41}b_{21}c_4 \\ & - a_{41}^2b_{21}c_3 - a_{31}^2b_{41}^2 + a_{31}a_{41}b_{31}b_{41} + a_{31}a_{41}b_{31}b_{41} - a_{41}^2b_{31}^2. \end{aligned} \tag{A13}$$

$$\begin{aligned} B = {} & c_3d_4(a_{11}b_{22} + a_{21}b_{21}) - c_4d_3(a_{11}b_{22} + a_{21}b_{21}) - a_{21}^2c_3d_4 + a_{21}^2c_4d_3 \\ & - b_{31}^2d_4(a_{11} - a_{12}) + b_{31}b_{41}d_3(a_{11} - a_{12}) + a_{21}a_{31}b_{31}d_4 - a_{21}a_{31}b_{41}d_3 \\ & + (a_{11} - a_{12})c_4b_{31}b_{41} - c_3b_{41}^2(a_{11} - a_{12}) - c_4a_{21}a_{41}b_{31} + c_3a_{21}a_{41}b_{41} \\ & + a_{21}a_{31}b_{31}d_4 - a_{21}a_{41}b_{31}d_3 - a_{31}a_{31}b_{22}d_4 + a_{31}a_{41}b_{22}d_3 - a_{21}a_{31}b_{41}c_4 \\ & + a_{21}a_{41}b_{41}c_3 + a_{31}a_{41}b_{22}c_4 - a_{41}^2b_{22}c_3 + a_{31}^2b_{41}^2 - a_{31}a_{41}b_{31}b_{41} \\ & - a_{31}a_{41}b_{31}b_{41} + a_{41}^2b_{41}^2. \end{aligned} \tag{A14}$$

$$C = a_{12}b_{22}c_3d_4 - a_{12}b_{22}c_4d_3 - a_{12}d_4b_{31}^2 + a_{12}b_{31}b_{41}d_3 - c_3a_{12}b_{41}^2 + c_4b_{31}b_{41}a_{12}. \tag{A15}$$

$$N^{B}_{11} = b_{21}c_3d_4 - b_{21}c_4^2 - 2b_{41}b_{31}c_4 + b_{41}^2c_3 + d_4b_{31}^2. \tag{A16}$$

$$N^{C}_{11} = b_{22}c_3d_4 - b_{22}c_4^2 + 2b_{41}b_{31}c_4 - b_{41}^2c_3 - d_4b_{31}^2. \tag{A17}$$

$$N^{B}_{12} = a_{41}b_{41}c_3 + a_{21}c_4^2 + d_4a_{31}b_{31} - a_{21}c_3d_4 - a_{41}b_{31}c_4 - a_{31}b_{41}c_4. \tag{A18}$$

$$N^{A}_{13} = a_{41}c_4b_{21} - d_4a_{31}b_{21} - a_{21}b_{31}d_4 - a_{31}b_{41}^2 + a_{21}b_{41}c_4 + a_{41}b_{31}b_{41}. \tag{A19}$$

$$N^{B}_{13} = a_{21}b_{31}d_4 + a_{31}b_{41}^2 - a_{21}b_{41}c_4 + a_{41}c_4b_{22} - d_4a_{31}b_{22} - a_{41}b_{31}b_{41}. \tag{A20}$$

$$N^{A}_{14} = b_{21}(c_4a_{31} - a_{41}c_3) - a_{41}b_{31}^2 - a_{21}b_{41}c_3 + a_{21}b_{31}c_4 + b_{31}b_{41}a_{31}. \tag{A21}$$

$$N^{B}_{14} = a_{41}b_{31}^2 + a_{21}b_{41}c_3 - a_{21}b_{31}c_4 - b_{31}b_{41}a_{31} + b_{22}(c_4a_{31} - a_{41}c_3). \tag{A22}$$

$$N^{A}_{22} = a_{11}c_3d_4 - d_4a_{31}^2 + 2a_{41}a_{31}c_4 - a_{11}c_4^2 - a_{41}^2c_3. \tag{A23}$$

$$N^{B}_{22} = a_{12}c_3d_4 - a_{12}c_4^2. \tag{A24}$$

$$N^{B}_{23} = a_{11}b_{41}c_4 + d_4a_{21}a_{31} - a_{11}b_{31}d_4 - a_{41}a_{21}c_4 - a_{41}a_{31}b_{41} + a_{41}^2b_{31}. \tag{A25}$$

$$N^{C}_{23} = a_{12}b_{41}c_4 - a_{12}b_{31}d_4. \tag{A26}$$

$$N^{A}_{24} = a_{11}b_{31}c_4 + a_{41}a_{21}c_3 + b_{41}a_{31}^2 - a_{31}a_{41}b_{31} - a_{11}b_{41}c_3 - c_4a_{21}a_{31}. \tag{A27}$$

$$N^{B}_{24} = a_{12}b_{31}c_4 - a_{12}b_{41}c_3. \tag{A28}$$

$$N^{A}_{33} = a_{11}b_{21}d_4 - 2a_{41}a_{21}b_{41} + a_{11}b_{41}^2 + d_4a_{21}^2 - b_{21}a_{41}^2. \tag{A29}$$

$$N^{B}_{33} = a_{11}b_{22}d_4 + a_{12}b_{21}d_4 + 2a_{41}a_{21}b_{41} - a_{11}b_{41}^2 - d_4a_{21}^2 - b_{22}a_{41}^2 + a_{12}b_{41}^2. \tag{A30}$$

$$N^{C}_{33} = a_{12}b_{22}d_4 - a_{12}b_{41}^2. \tag{A31}$$

$$N_{34}^{A} = a_3^1 a_{41} b_{21} - a_{11} b_{21} c_4 - a_{11} b_{31} b_{41} - a_{21}^2 c_4 + a_{41} a_{21} b_{31} + a_{21} a_{31} b_{41}, \tag{A32}$$

$$N_{34}^{B} = a_{31} a_{41} b_{22} + a_{11} b_{31} b_{41} + a_{21}^2 c_4 - a_{41} a_{21} b_{31} - a_{21} a_{31} b_{41} - a_{12} b_{31} b_{41} - c_4 a_{11} b_{22} - c_4 a_{12} b_{21}, \tag{A33}$$

$$N_{34}^{C} = a_{12} b_{31} b_{41} - c_4 a_{12} b_{22}, \tag{A34}$$

$$N_{44}^{A} = a_{11} b_{21} c_3 - b_{21} a_{31}^2 - 2 a_{31} a_{21} b_{31} + a_{11} b_{31}^2 + c_3 a_{21}^2, \tag{A35}$$

$$N_{44}^{B} = c_3 a_{11} b_{22} + c_3 a_{12} b_{21} + 2 a_{31} a_{21} b_{31} - a_{11} b_{31}^2 - c_3 a_{21}^2 - b_{22} a_{31}^2 + a_{12} b_{31}^2, \tag{A36}$$

$$N_{44}^{C} = a_{12} b_{22} c_3 - a_{12} b_{31}^2, \tag{A37}$$

and

$$a_{11} = \beta^2 (1 - \omega_3^2)(C_{11}^0 - C_{66}^0), \tag{A38}$$

$$a_{12} = \beta^2 (1 - \omega_3^2) C_{66}^0 + C_{55}^0 \omega_3^2, \tag{A39}$$

$$a_{21} = \beta^2 (1 - \omega_3^2)(C_{12}^0 + C_{66}^0), \tag{A40}$$

$$a_{31} = \beta \omega_3 (1 - \omega_3^2)^{1/2} (C_{13}^0 + C_{55}^0), \tag{A41}$$

$$a_{41} = \beta \omega_3 (1 - \omega_3^2)^{1/2} (e_{31}^0 + e_{15}^0), \tag{A42}$$

$$b_{21} = \beta^2 (1 - \omega_3^2)(C_{66}^0 - C_{11}^0), \tag{A43}$$

$$b_{22} = \beta^2 (1 - \omega_3^2) C_{11}^0 + C_{55}^0 \omega^2, \tag{A44}$$

$$b_{31} = \beta \omega_3 (1 - \omega_3^2)^{1/2} (C_{23}^0 + C_{44}^0), \tag{A45}$$

$$b_{41} = \beta \omega_3 (1 - \omega_3^2)^{1/2} (e_{31}^0 + e_{15}^0), \tag{A46}$$

$$C_3 = \beta^2 (1 - \omega_3^2) C_{55}^0 + C_{33}^0 \omega_3^2, \tag{A47}$$

$$C_4 = \beta^2 (1 - \omega_3^2) e_{15}^0 + e_{33}^0 \omega_3^2, \tag{A48}$$

$$d_4 = -\beta^2 (1 - \omega_3^2) \alpha_{11}^0 - \omega_3^2 \alpha_{33}^0, \tag{A49}$$

where β is the aspect ratio of the spheroidal inclusion. The integrals with respect to θ in eqns (A1)–(A12) can be obtained by the residue calculation in a complex Z-plane, where

$$\cos\theta = (z + z^{-1})/2, \quad \sin\theta = (z - z^{-1})/(2i), \quad d\theta = dz/iz. \tag{A50}$$